

## SELF-AVOIDING RANDOM MANIFOLDS \*

François DAVID<sup>†</sup>Service de Physique Théorique<sup>‡</sup>, CEA Saclay,

F-91191 Gif-sur-Yvette CEDEX, FRANCE

## 1 Tethered Surfaces and Random Manifolds

Several important developments of theoretical physics in the last 15 years come from the extension of the concept of random walk to fluctuating extended objects. This has been very fruitful both in high energy physics, where the quantum fluctuations of strings (1 + 1-dimensional objects) and of  $p$ -branes ( $p + 1$ -dimensional) in Minkowski space are considered, and in condensed matter physics, where the thermal fluctuations of 2-dimensional films or membranes in Euclidean 3-dimensional space are a fascinating subject (see for instance [1]).

It is known that a 2-dimensional surface, with an intrinsic metric  $g_{\alpha\beta}$ , embedded in flat  $d$ -dimensional target space (the embedding being described by the mapping  $x \rightarrow \vec{r}(x)$ ) is characterized by its extrinsic metric

$$h_{\alpha\beta} = \partial_\alpha \vec{r} \partial_\beta \vec{r} \quad (1)$$

Dimensional analysis shows that the relevant terms involving  $\vec{r}$  in the action may involve one derivative  $\partial_\alpha \vec{r}$  (tangent vectors) and two derivatives  $\partial_{\alpha\beta} \vec{r}$  (extrinsic curvatures) of the embedded surface. Two very different classes of models exist, characterized by the coupling between the intrinsic and the extrinsic metric.

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<sup>†</sup>Physique Théorique CNRS

<sup>‡</sup>Laboratoire de la Direction des Sciences de la Matière du CEA

(1) The intrinsic metric is proportional to the extrinsic one, i.e. is coupled to the embedding by the constraint  $g_{\alpha\beta} = \partial_\alpha \vec{r} \partial_\beta \vec{r}$ . This is the case for the “rigid string” model of Polyakov, and for the equivalent Canham-Helfrich model of fluid membranes with bending rigidity. The action is

$$S[\vec{r}] = \int d^2x \sqrt{g} \left[ \tau + \frac{\kappa}{2} (\Delta \vec{r})^2 \right] \quad (2)$$

with  $\tau$  the bare string (or surface) tension, and  $\kappa$  the bending rigidity. Renormalization group calculations show that at large distances, the bending rigidity becomes irrelevant, and that the effective action is nothing but that of the Polyakov string, with an effective string tension  $\tau_{\text{eff}}$  dynamically generated [2, 3].

(2) The intrinsic metric  $g_{\alpha\beta}$  does not fluctuate (or it has a dynamics decoupled from that of the extrinsic metric). This class of models is not very useful for high energy physics, but is relevant in statistical physics to describe the so-called “tethered membranes”, which generalize the concept of flexible 1-dimensional chains (polymers) to 2-dimensional networks [4]. It is this class of models that I shall discuss in these lectures.

Let me consider a flexible two dimensional regular triangular network fluctuating in 3-dimensional space. A simple discrete action for the model is

$$S = -\kappa \sum_{\substack{\text{neighbouring} \\ \text{triangles } t, t'}} \vec{n}_t \cdot \vec{n}_{t'} + \tau \sum_{\text{links } i, j} (\vec{r}_i - \vec{r}_j)^2 \quad (3)$$

where  $\vec{n}_t$  is the normal vector to the triangle  $t$ . The first term is the extrinsic curvature term, with  $\kappa$  the bending rigidity, the second term is a Gaussian elastic term. Numerical simulations and analytical arguments indicate a very interesting behavior for such a model. If  $\kappa$  is small enough, the bending rigidity is irrelevant at large distances, and the surface is in a “crumpled”, or “collapsed”, phase, with  $\langle \vec{n}_t \rangle = 0$ . This means that O(3) rotational invariance is not broken and that the surface has no average orientation. If  $\kappa$  is large enough, the surface is in a flat phase, characterized by a non-zero average orientation  $\langle \vec{n} \rangle \neq 0$ , and a spontaneous breakdown of O(3) invariance. These two phases are separated by a crumpling transition at some  $\kappa_c$ . Numerical simulations indicate that (at  $d = 3$ ) this transition is continuous, and characterized by non-trivial critical exponents.

An continuous model à la Landau-Ginzburg to describe tethered surfaces is given by the effective action [5]

$$S = \int d^D x \left[ \frac{\kappa}{2} (\Delta \vec{r})^2 + \frac{t}{2} h_{\alpha\alpha} + \frac{K}{2} (h_{\alpha\alpha})^2 + \mu \left( h_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{D} h_{\gamma\gamma} \right)^2 \right] \quad (4)$$

with  $h_{\alpha\beta} = \partial_\alpha \vec{r} \partial_\beta \vec{r}$  the extrinsic metric.  $D$  is the internal dimension of the surface (the model describes in fact  $D$ -dimensional tethered manifolds),  $\Delta = \partial_\alpha \partial_\alpha$  is the Laplacian,  $\kappa$  the bending rigidity.  $t$ ,  $K$  and  $\mu$  are the effective elastic moduli ( $t$  corresponds to a tension,  $K$  and  $\mu$  are related to the so-called Lamé coefficients). Neglecting the fluctuations, the minimization of this action shows that if  $t > 0$ ,  $\langle \partial_\alpha \vec{r} \rangle = 0$  and  $\langle h_{\alpha\beta} \rangle = 0$ , so that the surface is crumpled, while if  $t < 0$   $\langle \partial_\alpha \vec{r} \rangle \neq 0$  and the surface is flat.  $\langle h_{\alpha\beta} \rangle$  vanishes and there is a continuous crumpling transition at  $t = 0$ . This analysis neglects the effect of fluctuations and is valid only if the internal dimension of the manifold is  $D > 4$ . For  $D < 4$  fluctuations become important, and their effect can be estimated by an  $\epsilon$ -expansion for  $D = 4 - \epsilon$  [5], or by a large  $d$  expansion [6, 7, 8] (where  $d$  is the dimension of the target space). It might seem surprising that for 2-dimensional manifolds ( $D = 2$ ) the flat phase still exists, since it is characterized by a spontaneous breakdown of the continuous rotational  $O(d)$  symmetry, which should be forbidden by the Mermin-Wagner-Coleman theorem. Such a crumpling transition is indeed forbidden for  $D = 1$  (polymers): for any dimension  $d \geq 1$  of target space, infinite semi-flexible polymers with a non-zero bending rigidity  $\kappa$  are always crumpled at large distance, and the flat phase does not exist. In fact there is no contradiction for  $D = 2$ . The transverse degrees of freedom of the manifolds  $\vec{r}_\perp$  (undulations) are coupled to the longitudinal degrees of freedom  $\vec{r}_\parallel$  (phonons), so that the global symmetry of the model is not the compact group  $O(d)$  (rotations), but the non-compact group of Euclidean displacements  $E(d)$  (translations+rotations). There is a non-trivial coupling between phonons (longitudinal modes) and undulations (transverse modes) which generates effective long range interactions between these transverse modes. In the presence of such long range interactions the Mermin-Wagner theorem does not apply.

Let me give a tentative picture which emerges from the analytical and numerical studies of the crumpling transition. For  $D > 4$  the crumpling transition is continuous and its critical exponents are given by mean field

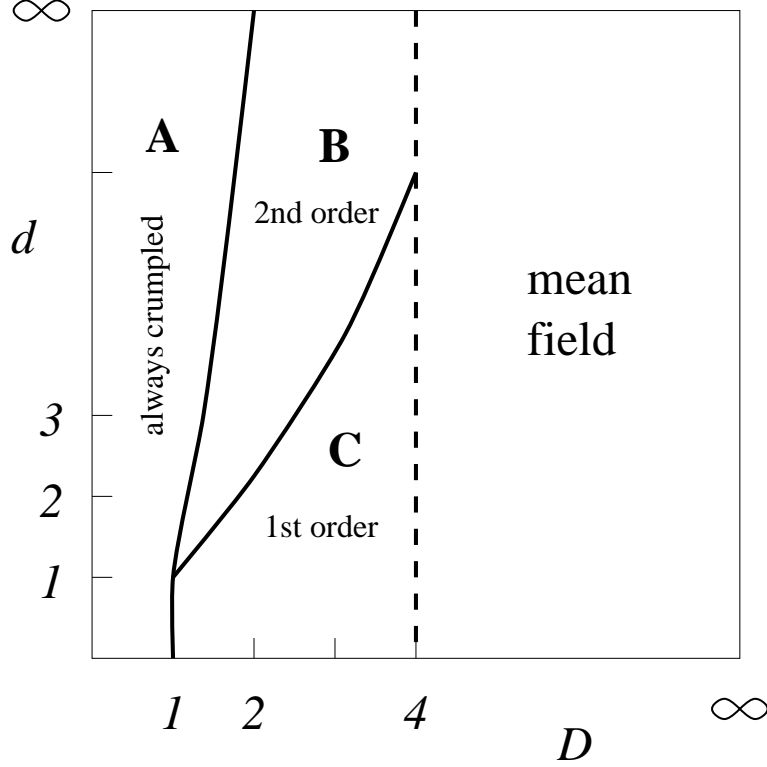


Figure 1: Nature of the crumpling transition for phantom manifolds as a function of  $D$  and  $d$

theory. For  $D = 4 - \epsilon$ ,  $\epsilon$  small, one loop calculations indicates that the crumpling transition is continuous for  $d > d_c$  large enough, but becomes discontinuous (fluctuation induced first-order) for  $d < d_c$  small. Large  $d$  calculations shows that there is a second order crumpling transition for  $D \geq D_1(d) < 2$  with  $D_1(\infty) = 2$ . For  $D < D_1(d)$  there is no flat phase and no crumpling transition. Thus we expect that the domain  $1 \geq D \geq 4$  in the  $(d, D)$  plane will be separated into three regions. A domain **A** for small  $D$  where the manifold is always crumpled, a domain **B** where there is a second order crumpling transition, and a domain **C** where the crumpling transition is always first order. Numerical simulations indicates that the point  $(D = 2, d = 3)$  is in **B** [9]. Recent studies of the folding problem show that the point  $(D = 2, d = 2)$  is in **C** [10].

The crumpled phase is the simplest to characterize. It corresponds to

$t > 0$  in (4), and at large distance only the quadratic  $t$  term is relevant. Therefore the action for a crumpled manifold is Gaussian, and is nothing but the massless free field action

$$S = \int d^D x \frac{1}{2} (\nabla_x \vec{r})^2 \quad (5)$$

The properties of such manifolds are easy to compute. For instance, if one considers a finite manifold with an internal extent  $L$ , its average squared size  $\langle R^2 \rangle$  in target space scales as

$$\langle R^2 \rangle \propto \begin{cases} L^{(2-D)} & \text{if } D < 2 \\ \ln(L) & \text{if } D = 2 \\ \text{constant} & \text{if } D > 2 \end{cases} \quad (6)$$

which implies that the fractal dimension of a crumpled Gaussian manifold is  $d_f = 2D/(2 - D)$  if  $D < 2$ , and is infinite if  $D \geq 2$ .

## 2 Self-avoiding crumpled Manifolds

The above considerations apply to “phantom manifolds”, which are free to intersect themselves. Indeed, the action (4) takes into account only local couplings in the internal space. Such local couplings are the only relevant one for strings, but for physical tethered networks self-avoiding interactions, which involve elements of the manifold which are close in target space but are arbitrarily far apart in internal space, are relevant. It is expected that such interactions will change the scaling properties of the manifold in the crumpled phase and at the crumpling transition. For instance, in the crumpled phase, the average squared size will now scale as

$$\langle R^2 \rangle \propto L^{2\nu} \quad 0 < \nu < 1 \quad (7)$$

with a critical exponent  $\nu \geq \nu_0 = \sup[(2 - D)/2, 0]$ . If  $\nu_0 < \nu < 1$  the manifold will be swollen by self-avoidance, but still crumpled. This is the case for polymers  $D = 1$  for  $1 < d < 4$ , with for instance  $\nu = 3/4$  for  $d = 2$ . The effect of self-avoidance may even be so strong that the manifold stays flat and that the crumpling transition disappear, in this case  $\nu = 1$ . Such a behavior has been observed in several numerical simulations of self-avoiding tethered membranes in three dimensions.

A simple analytical model to describe crumpled self-avoiding manifolds has been introduced in [11, 12]. It is a simple extension of the continuous Edwards model for polymers. The action  $S$  (the free energy for a configuration  $\vec{r}$ ) is the sum of a Gaussian elastic energy and of a 2-body repulsive interaction, proportional to the coupling constant  $b$ :

$$S[\vec{r}] = \int d^D x \frac{1}{2} (\nabla_x \vec{r})^2 + b \int d^D x \int d^D y \delta^d(\vec{r}(x) - \vec{r}(y)) . \quad (8)$$

The internal dimension  $D$  may be taken as a continuous parameter, interpolating between polymers ( $D = 1$ ) and membranes ( $D = 2$ ). The issue is to compute the critical exponents describing the scaling behavior of large manifolds, for instance the exponent  $\nu$  (related to the fractal dimension  $d_f$  of the manifold by  $\nu = D/d_f$ ), and the configuration exponent  $\gamma$ , related to the scaling of the partition function  $Z$  of a finite manifold with internal extent  $L$  by

$$Z \propto L^{\gamma-1} \text{ constant}^{L^D} . \quad (9)$$

The mean field exponents are obtained by setting  $b = 0$ . One recovers the free Gaussian action and the exponents  $\nu_0 = (2 - D)/2$  and  $\gamma_0 = 1 - d(2 - D)/2$  (if  $D$  is not integer).

Dimensional analysis shows that the mean field theory is invalid if the engineering dimension of  $b$ ,  $\epsilon$ , is positive

$$[b] = \epsilon = 2D - d(2 - D)/2 > 0 . \quad (10)$$

In this case, we expect that  $\nu > \nu_0$ . For small enough  $d$ , and certainly for  $d \leq D$ , we expect that the manifold is flat. The general picture of the expected behavior as a function of the internal dimension  $D$  and of the external dimension  $d$  is presented in Fig. 2

A natural idea is to compute the corrections to mean field by a  $\epsilon$ -expansion à la Wilson-Fisher. This has been done for polymers, first by using the so-called de Gennes trick [13]: in the scaling limit the self-avoiding walk can be mapped (by a Legendre transform) onto a local field theory with  $O(n)$  symmetry defined in the  $d$ -dimensional target space, in the limit  $n \rightarrow 0$ . It is then equivalent to study the limit of a single very long polymer and the massless limit of the  $n = 0$  theory, for which standard renormalization group theory is applicable. Unfortunately no such equivalence exists for manifolds, beyond the case  $D = 1$ . Another renormalization scheme used

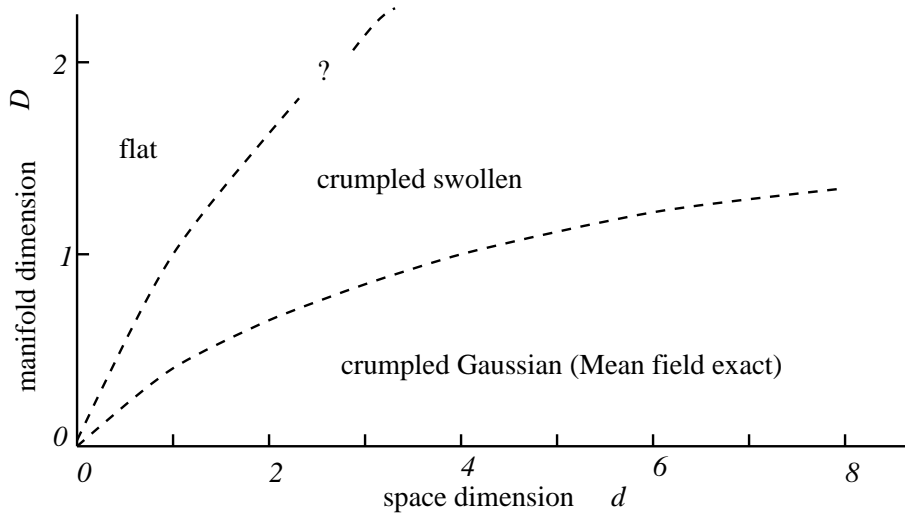


Figure 2: Self-avoiding manifolds as a function of  $d$  and  $D$

for polymers is the so-called direct renormalization scheme [14]. It has been applied to self-avoiding manifolds by Aronowitz & Lubensky [11] and by Kardar & Nelson [12], who performed calculations to first order in  $\epsilon$ . The basic idea of this method is to perform explicit perturbative calculations for a finite manifold and for  $\epsilon > 0$ . Perturbation theory is then UV and IR finite, but has UV divergences when  $\epsilon \rightarrow 0$ . It appears that these poles in  $1/\epsilon$  can be removed by reexpressing the observables in terms of adequate dimensionless renormalized quantities, such as the second virial coefficient. The internal size  $L$  of the manifold plays the role of the inverse of a renormalization mass scale, and renormalization group equations can be obtained by considering the  $L$  dependence of the renormalized theory. At first order in perturbation theory, the consistency of these renormalization group equations has been checked explicitly by Duplantier, Hwa & Kardar [15]. However, at that time it was not clear whether this direct renormalization approach could be justified beyond first order (except for  $D = 1$ , where the de Gennes trick is used to show the equivalence between direct renormalization and the standard minimal subtraction scheme).

### 3 Renormalization for multi-local Theories:

Recently it became possible to prove the consistency of this approach, and the renormalizability of the model (8) directly in the internal  $D$ -dimensional space, despite the fact that this model is a non-local field theory in  $D$  dimensions [16]. Let me present the general idea for the proof, which is due to B. Duplantier, E. Guitter and myself. The perturbation theory for this model is obtained by expanding the observables as power series in  $b$ . The bi-local “interaction vertex” (in field theoretic language I call it a bilocal operator) is written in Fourier transform as

$$\begin{array}{c} \bullet \text{---} \bullet \\ x \qquad y \end{array} = \delta^d(\vec{r}(x) - \vec{r}(y)) = \int d^d k \, e^{i\vec{k}(\vec{r}(x) - \vec{r}(y))} . \quad (11)$$

It can be viewed in a Coulomb gas representation as the integral over the “charge”  $\vec{k}$  of a neutral “dipole” with charge  $+\vec{k}$  at  $x$  and charge  $-\vec{k}$  at  $y$ . The term of order  $b^K$  in the perturbative expansion of the partition function (as well as of other observables) involves  $K$  dipoles  $(x_1, y_1), \dots, (x_K, y_K)$ . The integration over the charges  $\vec{k}_1, \dots, \vec{k}_K$  gives an integral over the positions of the dipoles of the determinant of the “dipole energy” quadratic form  $Q$

$$\int \cdots \int \prod_{i=1}^K d^D x_i d^D y_i \det [Q[x_i, y_i]]^{-d/2} . \quad (12)$$

$Q$  is a  $K \times K$  matrix such that  $\sum_{i,j=1}^K \vec{k}_i Q_{ij} \vec{k}_j$  is the Coulomb energy (in  $D$ -dimensions) of the  $K$  dipoles. Each  $Q_{ij}$  is a linear combination

$$Q_{ij} = G_0(x_i, x_j) + G_0(y_i, y_j) - G_0(x_i, y_j) - G_0(x_j, y_i) . \quad (13)$$

of the Coulomb potentials  $G_0$  between the endpoints of pairs of dipoles  $i$  and  $j$

$$G_0(x, x') = \langle r(x)r(x') \rangle_0 = \frac{\Gamma((D-2)/2)}{4\pi^{D/2}} |x - x'|^{2-D} . \quad (14)$$

This Coulomb potential is properly defined for  $0 < D < 2$  by analytic continuation in  $D$ . For  $0 < D < 2$  it is negative, but it vanishes for  $x = x'$  (while for  $D > 2$  it diverges), and it decreases at large distances (as for  $D > 2$ ). The integration over the  $2K$  points in a non-integer  $D$ -dimensional space



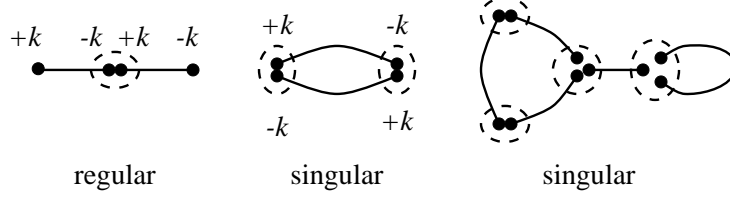


Figure 3: A non-neutral regular configuration and two neutral UV singular configurations

can also be defined properly by analytic continuation in  $D$  and the use of distance geometry. This amounts to replace the integration over the  $2K \times D$  coordinates of the  $2K$  points by an integration over the  $K \times (2K - 1)$  scalar distances between these points.

One can show that when the determinant  $\det[Q]$  vanishes short distance UV singularities occur in the integrals. This occurs if and only if some of the end-points of (not necessarily the same) dipoles coincide, so that the end-points form “atoms”, while the dipoles form “molecules”, and if moreover one can assign non-zero charges  $\vec{k}_i$  to the dipoles while each atom stays globally neutral. This condition is more easily depicted graphically on Figure 3.

The associated singularities of these integrals are related to the behavior at short distance of the expectation value (with respects to the free Gaussian model) of products of bilocal operators as given by Equ. 11. One can show that this short distance behavior is encoded in a multilocal operator product expansion (MOPE) , which generalizes Wilson’s operator product expansion. Let me give two examples:

When the two points  $x$  and  $y$  of the bi-local interaction operator tend towards a single point, this operator can be expanded in terms of local operators involving derivatives of the field  $\vec{r}$ . The first terms of the expansion are explicitly (not writing explicitly the  $D$  and  $d$  dependence of the coefficients)

$$\begin{aligned}
 \text{Diagram: A loop with two dots labeled } x \text{ and } y \text{ at its base.} &= c_0 |x - y|^{\epsilon - 2D} \mathbf{1} \\
 &+ c_1 |x - y|^{\epsilon - D - 2} (x^\alpha - y^\alpha)(x^\beta - y^\beta) : \nabla_\alpha \vec{r} \nabla_\beta \vec{r} : \\
 &+ \dots
 \end{aligned} \tag{15}$$

$\mathbf{1}$  is the identity operator (its expectation value is 1), the  $:$  in the operator

$:\nabla_\alpha\vec{r}\nabla_\beta\vec{r}:$  denotes the normal ordering subtraction prescription required to deal properly with the UV singularities contained in  $\nabla_\alpha\vec{r}\nabla_\beta\vec{r}$ .

The second example is less simple, and shows that when the end-points of two bilocal operators tend pairwise towards two different points, this generates again bilocal operators

$$\begin{array}{c} x_1 \quad \text{---} \quad \text{---} \quad y_1 \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ x_2 \quad \text{---} \quad \text{---} \quad y_2 \end{array} = d_0 \left[ |x_1 - x_2|^{2-D} + |y_1 - y_2|^{2-D} \right]^{-d/2} \text{---} \text{---} + \dots \quad (16)$$

This structure is generic, and products of local and bilocal operators generate multilocal operators of the general form

$$\Phi\{x_1, \dots, x_P\} = \int d^d\vec{r}_0 \prod_{i=1}^P \left[ (\nabla_{\vec{r}_0})^{m_i} \delta^d(\vec{r}_0 - \vec{r}(x_i)) A_i(x_i) \right] . \quad (17)$$

where the  $A_i(x_i)$  are local operators, which can be decomposed into products of multiple  $x$ -derivatives of  $\vec{r}$ . The  $m_i$  are integers. For  $P = 1$  and  $m = 0$  one recovers local operators  $A(x)$  ( $m > 0$  gives 0). For  $P = 2$ ,  $m_1 = m_2 = 0$  and  $A_1 = A_2 = \mathbf{1}$  one recovers the bilocal interaction operator, etc... These operators have a very special form: they can be viewed as a local convolution in the target  $d$ -dimensional  $\vec{r}$  space of a non-local product (in the internal  $D$ -dimensional space) of the  $P$  local operators  $A_i$ .

The MOPE implies that the formalism of renormalization theory and of renormalization group equations, which has been developed for local quantum field theories, can be adapted for this model. One is in fact interested in the IR scaling behavior of the lattice model, when some length scale  $L$  goes to  $\infty$ . This lattice model is described by the action (8), with a short distance lattice cut-off  $a$ . To study this IR limit it is equivalent to look at the UV continuum limit of the model when the physical length scale  $L$  is kept fixed, while the UV cut-off  $a$  goes to 0. In this limit one can construct, via renormalization, a finite renormalized theory with  $a = 0$ , which obeys renormalization group equations. From these equations, one recovers the large distance behavior of the lattice model we started from. The procedure works well in perturbation theory when one is close to the upper critical dimension, i.e. for  $\epsilon$  small, and it leads to the  $\epsilon$ -expansion.

In our case, the MOPE can be used to determine, by power counting, which multilocal operators are relevant and give UV singularities (poles in

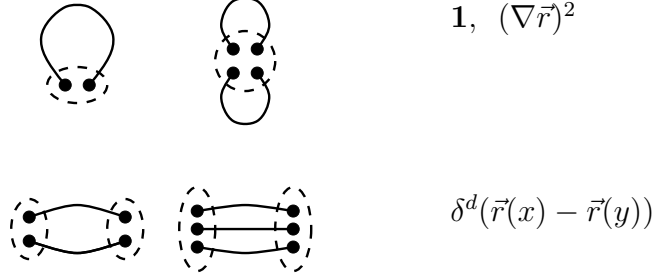


Figure 4: UV divergent configurations and the associated relevant operators

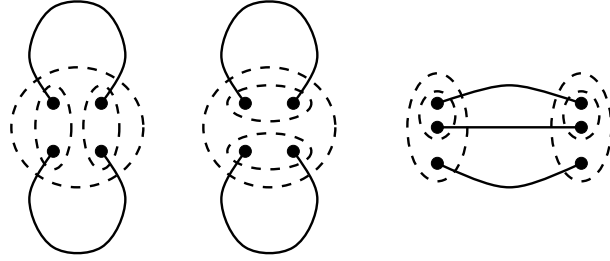


Figure 5: Examples of nested singular configurations

$1/\epsilon$ ). Then one can also show that these poles can be subtracted by adding to the action (8) counterterms proportional to the marginally relevant multi-local operators, leading to the UV finite renormalized theory. For the model of self-avoiding manifolds, this analysis shows that the UV divergences are associated only with local and bilocal operators, as depicted on Fig. 4, and that only three operators are relevant: the identity operator  $\mathbf{1}$ , the elastic energy operator  $(\nabla \vec{r})^2$  and the bilocal operator  $\delta^d(\vec{r}(x) - \vec{r}(y))$ .  $\mathbf{1}$  is strongly relevant, and gives power-like UV divergences proportional to  $a^{-D}$  ( $a$  being a short-distance cut-off). The two other operators are superficially relevant, they give logarithmic UV divergences or equivalently poles in  $1/\epsilon$  at  $\epsilon = 0$ .

The fact that the so-called superficial divergences, associated to a global contraction of points towards a singular configuration, can be subtracted by counterterms is a consequence of the MOPE. A complete proof of the renormalizability of the theory is possible, but much more delicate. It requires a control of the subdivergences coming from successive contractions associated to nested singular configurations, such as those depicted of Fig. 3 .

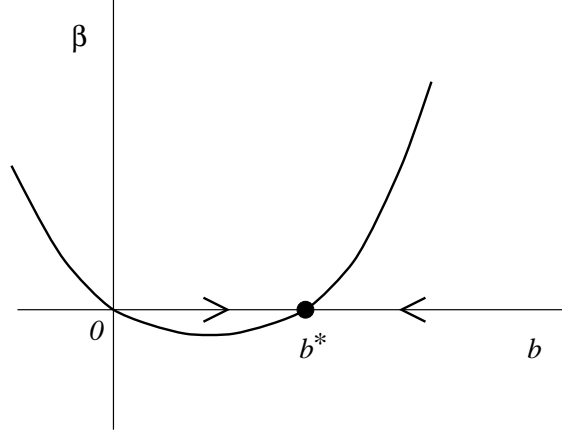


Figure 6: The  $\beta$ -function and the RG flow for  $\epsilon > 0$ .

## 4 Scaling for infinite self-avoiding Manifold

A first application of this formalism is the derivation of scaling laws. Since the model is renormalizable (at least perturbatively), it can be made UV finite (for  $\epsilon \simeq 0$ ) by introducing two counterterms in the action. The new renormalized action is of the form

$$S[\vec{r}] = \frac{Z}{2} \int d^D x (\nabla \vec{r})^2 + b_R \mu^\epsilon Z_b \int \int d^D x d^D y \delta(\vec{r}(x) - \vec{r}(y)) . \quad (18)$$

$b_R$  is the dimensionless renormalized coupling constant (the perturbative expansion in  $b_R$  is UV finite order by order).  $Z$  is a wave-function renormalization factor and  $Z_b$  a coupling constant renormalization factor, both are perturbative series in  $b_R$ , with poles up to degree  $1/\epsilon^{K-1}$  at order  $K$ .  $\mu$  is the renormalization momentum scale. As for ordinary local theories, such as the Landau-Ginzburg-Wilson  $\Phi^4$  model, one can change  $b_R$  and  $\vec{r}$  in Equ. 18 into bare quantities in order to rewrite the renormalized Hamiltonian as a bare action given by Equ. 8. The renormalization group  $\beta$ -function and the anomalous dimension  $\gamma$  of the field  $\vec{r}$  are defined in the standard way

$$\beta(b_R) = \mu \frac{\partial}{\partial \mu} b_R \Big|_{\text{bare}} ; \quad \gamma(b_R) = -\frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z \Big|_{\text{bare}} . \quad (19)$$

The  $\beta$ -function is found to be of the form

$$\beta(b_R) = -\epsilon b_R + \mathbf{c} b_R + \mathcal{O}(b_R^2) \quad ; \quad \mathbf{c} = \mathbf{c}(D) \text{ positive constant} , \quad (20)$$

and therefore there is, at least for small  $\epsilon > 0$ , an IR attractive fixed point  $b_r^* = \mathcal{O}(\epsilon)$ , which governs the scaling behavior of self-avoiding polymerized surfaces at large distance. The existence of this fixed point ensures the universality of this non-trivial scaling for  $\epsilon > 0$ , and that no new interactions, possibly non-local in external space, are generated by the RG transformations.

The explicit calculation for the scaling exponents  $\nu$  and  $\gamma$  leads to the same results for the scaling exponents  $\nu$  and  $\gamma$  than the direct renormalization method at first order in  $\epsilon$ . With this method higher order calculations are feasible, but technically quite difficult. In particular, already at second order the RG functions cannot be expressed analytically, and numerical integration methods have to be developed. Work is in progress to compute the scaling exponents at order  $\epsilon^2$ .

## 5 Finite Size Scaling and direct Renormalization

The model given by Equ. 8 describes an infinite manifold with flat internal metric, corresponding to an infinite and regular flexible lattice. Finite manifolds are described by a similar model, but the  $D$ -dimensional manifold  $M$  is now embodied with a fixed non-trivial Riemannian metric  $g_{\alpha\beta}(x)$  (examples are closed manifolds with the topology of the sphere  $\mathcal{S}_D$  or the torus  $\mathcal{T}^D$ ), and may have a boundary  $\partial M$  (open manifold with the topology of the disk for instance). A similar analysis can be performed for such models, and the MOPE structure of short distance singularities is still valid, but new local operators  $A(x)$ , which depend on the internal metric on  $M$  and on the boundary  $\partial M$ , such as the scalar curvature  $R$ , appear in the MOPE and in Equ. (17). The renormalized action now contains at least five operators and five independent renormalization factors  $Z$

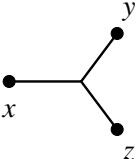
$$\begin{aligned} S[\vec{r}] = & \int_M Z \mathbf{1} + \int_M Z (\nabla \vec{r})^2 + \int \int_M Z b \delta^d(\vec{r} - \vec{r}) \\ & + \int_M Z R + \int_{\partial M} Z \mathbf{1} . \end{aligned} \quad (21)$$

The curvature operator  $\int_M R$  is superficially relevant only for  $D = 2$  and the boundary operator  $\int_{\partial M} \mathbf{1}$  only for  $D = 1$ . When these additional terms are not relevant, the first three renormalization factors  $Z$  are the same for finite curved manifolds than for the infinite flat plane. This property is analogous to the renormalization property of local field theories in finite geometries, which justifies the finite scaling laws for critical systems in finite geometries, and it has two very important consequences: (i) The scaling hypothesis at the basis of the direct renormalization approach, which relies explicitly on calculations with finite manifolds, is shown to be valid to all orders in perturbation theory; (ii) for “abstract” manifolds with dimension  $D < 2$ , with the only exception of open polymers ( $D = 1$  open surface), the following hyperscaling relation [17] relating the configuration and the  $\nu$  exponents holds:

$$\gamma = 1 - \nu d . \quad (22)$$

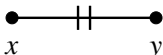
## 6 Self-avoiding Manifold at the tricritical $\Theta$ -point:

Finally, let me briefly discuss recent results obtained with K. Wiese on the scaling behavior of polymerized membranes at the  $\Theta$ -point [18]. This point separates the swollen phase, where the self-avoidance repulsive forces that I considered previously dominate, from the dense collapsed phase, where short ranged attractive forces dominate. At the  $\Theta$ -point the effective two body repulsive coupling  $b$  vanishes, and two different interactions may become relevant. The first one is the 3-body contact repulsion, which is usually considered for polymers



$$= \int \int \int d^D x d^D y d^D z \delta^d(\vec{r}(x) - \vec{r}(y)) \delta^d(\vec{r}(x) - \vec{r}(z)) . \quad (23)$$

The second one is a modified 2-body interaction, repulsive at short range but attractive at larger range ( $\Delta_{\vec{r}}$  is the  $d$ -dimensional Laplacian)



$$= - \int \int d^D x d^D y \Delta_{\vec{r}} \delta^d(\vec{r}(x) - \vec{r}(y)) . \quad (24)$$

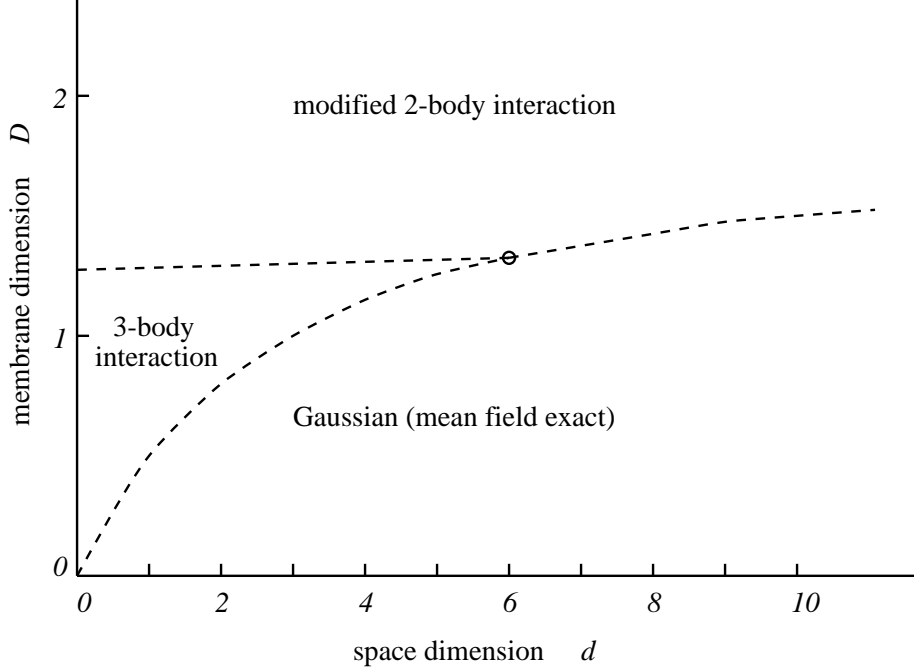


Figure 7: Relevant interactions for the  $\Theta$ -point in the  $d$ - $D$  plane.

Calculations at first order are not feasible analytically, and already require numerical evaluations of complicated integrals. The results of such one loop calculations are schematically depicted on Fig. 6, where the domains where the 3-body and modified 2-body terms are respectively relevant are shown. This indicates that the last modified 2-body term is the relevant one for 2-dimensional manifold in any external dimension  $d$ . There is also a quite interesting and non-trivial crossover between the two terms around  $D = 4/3$   $d = 6$ , which must be studied by a double  $\epsilon$ -expansion.

## 7 Conclusion:

The theoretical study of the scaling behavior of polymerized flexible membranes leads to the development of new multilocal continuum field theories, and to new applications of renormalization group methods. I hope that these methods will lead to a quantitative progress in the understanding of the behavior of real 2-dimensional polymerized membranes. This requires results

beyond first order in the  $\epsilon$ -expansion (recall that  $D = 2$  correspond to  $\epsilon = 4$ ), and a better understanding of the relation between this RG approach and more heuristic or approximate methods, such as variational methods or approximate recursion relations. The sophisticated renormalization theory for multilocal models presented here should hopefully also find applications in other problems of statistical physics, or in other areas of theoretical physics.

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